THE BOUNDARY VALUE PROBLEM FOR LAPLACIAN ON DIFFERENTIAL FORMS AND CONFORMALLY EINSTEIN INFINITY

MATTHIAS FISCHMANN AND PETR SOMBERG

Abstract. We completely resolve the boundary value problem for differential forms and conformally Einstein infinity in terms of the dual Hahn polynomials. Consequently, we produce explicit formulas for the Branson-Gover operators on Einstein manifolds and prove their representation as a product of second order operators. This leads to an explicit description of Q-curvature and gauge companion operators on differential forms.

Contents

1. Introduction

Boundary value problems have ever played an important role in mathematics and physics. A preferred class of boundary value problems is given by a system of partial differential equations on manifolds with boundary (or, a submanifold) equipped with a geometrical structure. The representative examples are the Laplace and Dirac operators on Riemannian manifolds with boundaries. An intimately related concept is the Poisson transform and boundary (or, submanifold) asymptotic of a solution for a system of PDEs, cf. [KKM+78] for the case related to compactifications of symmetric spaces.

In [FG11], Fefferman and Graham initiated a program allowing to regard a conformal manifold as the conformal infinity of associated Poincaré-Einstein metric. The boundary value problems on the Poincaré-Einstein manifolds for eigenvalue type of

2010 *Mathematics Subject Classification.* 53C21, 53C25, 53A30, 34L10, 33C20.

Key words and phrases. Boundary value problems, Einstein manifolds, Generalized hypergeometric functions, Conformal geometry, Branson-Gover operators, Q-curvature operators.

Research supported by grant GA CR P201/12/G028.

differential equations (e.g., Laplace, Dirac, etc.) with prescribed boundary data are referred to as the boundary value problems for conformal infinity. It is a remarkable fact that solving such a boundary value problems leads to an algorithmic (or, recursive) construction of a series of conformally covariant differential operators on functions, spinors and differential forms [GZ03, GMP10, AG08]. Note that these operators were originally constructed using the ambient metric of Fefferman and Graham and tractor bundles, cf. [GJMS92, HS01, BG05], and were soon recognized to encode interesting geometrical quantities like Branson's Q-curvature [Bra93] or holographic deformations of the Yamabe and Dirac operators [Juh13, Fis13]. Moreover, their functional determinants play a fundamental role in quantum field theories, [Ros87].

Let (M, h) be an Einstein manifold. The main result of the present article is the complete and explicit solution of the boundary value problem for the Laplace operator acting on differential forms and the conformally Einstein infinity (M, h). More precisely, we reduce the boundary value problem to a rank two matrix valued system of four step recurrence relations for the coefficients of the asymptotic expansion of form Laplace eigenforms. This combinatorial problem can be resolved in terms of generalized hypergeometric functions, closely related to the dual Hahn polynomials. The results analogous to ours were obtained for scalar and spinor fields in [FG11, FKS15]. The key property of reducing the boundary value problem for conformally Einstein infinity is the polynomial character of the 1-parameter family of metrics given by the Poincaré-Einstein metric. As an application, we recover explicit formulas for the Branson-Gover and related Q-curvature operators on differential forms on Einstein manifolds and produce their factorization as a product of second-order differential operators [GŠ13].

Let us briefly indicate the content of our article. The Section 2 is combinatorial in its origin with some implications to hypergeometric function theory. We introduce three series of polynomials $s_m^{(-)}$, $s_m^{(+)}$ and $s_m^{(1)}$ of degree $m \in \mathbb{N}_0$, depending on spectral parameters. Their origin is motivated by the examples given in Subsection 3.2. We prove that $s_m^{(\pm)}$ satisfy a three step recurrence relation, cf. Proposition 2.2 and 2.3, while $s_m^{(1)}$ turns out to be a linear combination of $s_k^{(-)}$ for $k = m, m-1, m-2, 0$, see Proposition 2.6. By a cascade of variable changes, we identify $s_m^{(\pm)}$ and $s_m^{(1)}$ as (a linear combination of) generalized hypergeometric functions, in particular $s_m^{(\pm)}$ are given by the dual Hahn polynomials.

In Section 3, we briefly recall the boundary value problem for conformal infinity. First of all, we determine in Proposition 3.2 its solution in terms of solution operators when the conformal infinity contains the flat metric. Then we prove in Proposition 3.6 that the polynomials $s_m^{(\pm)}$ and $s_m^{(1)}$ mentioned above are the organizing framework for the solution once the conformal infinity contains an Einstein metric.

In Section 4, we discuss the emergence of Branson-Gover operators in the framework of solution operators. Furthermore, we prove in Theorem 4.3 and Theorem 4.4 that Branson-Gover operators factorize by second-order differential operators. Finally, we discuss explicit formulas for the gauge companion and the Q-curvature operators.

In Appendices A and B we collect some standard notation, results and properties concerning generalized hypergeometric functions and Poincaré-Einstein metrics.

2. Some combinatorial identities

In the present section we discuss a class of special polynomials characterized to satisfy certain recurrence relations.

Let y be an abstract variable and define the set of polynomials $R_k(y;\alpha)$,

$$R_k(y;\alpha) := \prod_{l=1}^{k} \big[y - (\alpha - l)(\alpha - l + 1)\big], \tag{2.1}$$

of degree $k \in \mathbb{N}$ and depending on a parameter $\alpha \in \mathbb{R}$. Conventionally, we set $R_0(y;\alpha) := 1$.

Remark 2.1 We notice that some versions of the polynomials $R_k(y;0)$ have already appeared in the scalar and spinor boundary value problems, see [FG11, Chapter 7] and [FKS15].

Furthermore, we introduce the polynomials

$$\begin{aligned} s_m^{(-)}(y) &:= \sum_{k=0}^{m} \binom{m}{k} (\tfrac{\beta}{2} - \lambda - m)_{m-k} (-\tfrac{\beta}{2} - m - 1)_{m-k} R_k(y;0) \\ &=: \sum_{k=0}^{m} C_k^{(-)}(m) R_k(y;0), \end{aligned} \tag{2.2}$$

$$\begin{aligned} s_m^{(+)}(y) &:= \sum_{k=0}^{m} \binom{m}{k} (\tfrac{\beta}{2} - \lambda - m)_{m-k} (-\tfrac{\beta}{2} - m + 1)_{m-k} R_k(y;0) \\ &=: \sum_{k=0}^{m} C_k^{(+)}(m) R_k(y;0), \end{aligned} \tag{2.3}$$

for $m \in \mathbb{N}_0$ and two parameters $\beta, \lambda \in \mathbb{C}$. Here we already used the notion of Pochhammer symbol, as reviewed in Appendix A.

We shall now observe basic recurrence relations satisfied by $s_m^{(-)}$ and $s_m^{(+)}$.

Proposition 2.2 *The collection of polynomials $s_m^{(-)}$, $m \in \mathbb{N}$, satisfies the following recurrence relation*

$$\begin{aligned} s_m^{(-)}(y) = {}& \big[y + 2m(\lambda + m) + \tfrac{\beta}{2}(\lambda - \tfrac{\beta}{2} - 1)\big] s_{m-1}^{(-)}(y) \\ & - (m-1)(\lambda + m)(\lambda - \tfrac{\beta}{2} + m - 1)(\tfrac{\beta}{2} + m) s_{m-2}^{(-)}(y), \end{aligned} \tag{2.4}$$

with $s_0^{(-)}(y) = 1$, $s_{-1}^{(-)}(y) := 0$.

Proof. The identity

$$R_{k+1}(y;0) = \big[y - k(k+1)\big] R_k(y;0)$$

for $k \in \mathbb{N}$, leads to

$$\big[y + 2m(\lambda + m) + \tfrac{\beta}{2}(\lambda - \tfrac{\beta}{2} - 1)\big] s_{m-1}^{(-)}(y)$$

$$
\begin{aligned}
&= \sum_{k=0}^{m-1} C_k^{(-)}(m-1)\big[y - k(k+1)\big] R_k(y;0) \\
&\quad + \sum_{k=0}^{m-1} C_k^{(-)}(m-1)\big[(2m(\lambda+m) + \tfrac{\beta}{2}(\lambda - \tfrac{\beta}{2} - 1) + k(k+1)\big] R_k(y;0) \\
&= \sum_{k=1}^{m} C_{k-1}^{(-)}(m-1) R_k(y;0) \\
&\quad + \sum_{k=0}^{m-1} C_k^{(-)}(m-1)\big[(2m(\lambda+m) + \tfrac{\beta}{2}(\lambda - \tfrac{\beta}{2} - 1) + k(k+1)\big] R_k(y;0).
\end{aligned}
$$

Therefore, it remains to compare the coefficients by $R_k(y;0)$ on both sides of (2.4), which is equivalent to the following set of relations among $C_k^{(-)}(m)$:

$$
\begin{aligned}
C_m^{(-)}(m) &= C_{m-1}^{(-)}(m-1), \\
C_{m-1}^{(-)}(m) &= C_{m-2}^{(-)}(m-1) \\
&\quad + C_{m-1}^{(-)}(m-1)\big[(2m(\lambda+m) + \tfrac{\beta}{2}(\lambda - \tfrac{\beta}{2} - 1) + m(m-1)\big], \\
C_k^{(-)}(m) &= C_{k-1}^{(-)}(m-1) \\
&\quad + C_k^{(-)}(m-1)\big[(2m(\lambda+m) + \tfrac{\beta}{2}(\lambda - \tfrac{\beta}{2} - 1) + k(k+1)\big] \\
&\quad - C_k^{(-)}(m-2)\big[(m-1)(\lambda+m)(\lambda - \tfrac{\beta}{2} + m - 1)(\tfrac{\beta}{2} + m)\big],
\end{aligned}
$$

for all $k \in \mathbb{N}_0$ such that $k \leq m-2$, and $C_{-1}^{(-)}(m) := 0$ for all $m \in \mathbb{N}_0$. These relations can be easily verified using the identity

$$
C_r^{(-)}(m-l) = \frac{\binom{m-l}{r}(\frac{\beta}{2} - \lambda - m + l)_{m-l-r}(-\frac{\beta}{2} - m + l - 1)_{m-l-r}}{\binom{m}{k}(\frac{\beta}{2} - \lambda - m)_{m-k}(-\frac{\beta}{2} - m - 1)_{m-k}} C_k^{(-)}(m)
$$

with $l = 1, 2$ and $r = 0, \ldots, m-1$. This completes the proof. □

Proposition 2.3 *The collection of polynomials $s_m^{(+)}$, $m \in \mathbb{N}$, satisfies the recurrence relations*

$$
\begin{aligned}
s_m^{(+)}(y) &= \big[y + 2(m-1)(\lambda+m-1) + \tfrac{\beta}{2}(\lambda - \tfrac{\beta}{2} + 1)\big] s_{m-1}^{(+)}(y) \\
&\quad - (m-1)(\lambda+m-2)(\lambda - \tfrac{\beta}{2} + m - 1)(\tfrac{\beta}{2} + m - 2) s_{m-2}^{(+)}(y), \qquad (2.5)
\end{aligned}
$$

with $s_0^{(+)}(y) = 1$, $s_{-1}^{(+)}(y) := 0$.

Proof. It is completely analogous to the proof of the previous proposition. The claim is equivalent to

$$
\begin{aligned}
C_m^{(+)}(m) &= C_{m-1}^{(+)}(m-1), \\
C_{m-1}^{(+)}(m) &= C_{m-2}^{(+)}(m-1) \\
&\quad + C_{m-1}^{(+)}(m-1)\big[(2(m-1)(\lambda+m-1) + \tfrac{\beta}{2}(\lambda - \tfrac{\beta}{2} + 1) + m(m-1)\big],
\end{aligned}
$$

$$
\begin{aligned}
C_k^{(+)}(m) = C_{k-1}^{(+)}(m-1) \\
&+ C_k^{(+)}(m-1)\big[(2(m-1)(\lambda+m-1)+\tfrac{\beta}{2}(\lambda-\tfrac{\beta}{2}+1)+k(k+1)\big] \\
&- C_k^{(+)}(m-2)\big[(m-1)(\lambda+m-2)(\lambda-\tfrac{\beta}{2}+m-1)(\tfrac{\beta}{2}+m-2)\big],
\end{aligned}
$$

for $k \in \mathbb{N}_0$ such that $k \leq m-2$, and $C_{-1}^{(+)}(m) := 0$ for all $m \in \mathbb{N}_0$. However, these identities hold due to

$$
C_r^{(+)}(m-l) = \frac{\binom{m-l}{r}(\frac{\beta}{2}-\lambda-m+l)_{m-l-r}(-\frac{\beta}{2}-m+l+1)_{m-l-r}}{\binom{m}{k}(\frac{\beta}{2}-\lambda-m)_{m-k}(-\frac{\beta}{2}-m+1)_{m-k}} C_k^{(+)}(m)
$$

with $l = 1, 2$ and $r = 0, \dots, m-1$. This completes the proof. $\square$

Furthermore, we introduce another set of polynomials:

$$
R_k^{(1)}(y) := \sum_{j=1}^{k} (\tfrac{\beta}{2}-k+j+1)_{2(k-j)}\big[y-\tfrac{\beta}{2}(\tfrac{\beta}{2}+1)\big]R_{j-1}(y;k) \tag{2.6}
$$

of degree $k \in \mathbb{N}$, and set the convention $R_0^{(1)}(y) := 0$.

Proposition 2.4 *The polynomials $R_m(\cdot;0)$ and $R_m^{(1)}(\cdot)$ are related by*

$$
R_m(y;0) = R_m^{(1)}(y) + (\tfrac{\beta}{2}-m+1)_{2m}, \tag{2.7}
$$

for all $m \in \mathbb{N}_0$.

Proof. The left hand side is a polynomial in y of degree m, and so is the right hand side. Hence, it is sufficient to check that both sides of this polynomial identity have the same value at $m+1$ different points. To that aim, we choose the m-tuple $i(i-1)$, $i = 1, \dots, m$, of roots of $R_m(y;0)$. We note

$$
\begin{aligned}
R_m^{(1)}\big(i(i-1)\big) = \sum_{j=1}^{m}(-1)^j(\tfrac{\beta}{2}-m+j+1)_{2(m-j)}\times \\
&\times(\tfrac{\beta}{2}+i)(\tfrac{\beta}{2}-i+1)\prod_{l=1}^{j-1}\big[(m-l+i)(m-l-i+1)\big],
\end{aligned}
$$

and the standard combinatorial identities

$$
(\tfrac{\beta}{2}-m+j+1)_{2m-2j} = (-1)^j\frac{(\frac{\beta}{2}-m+1)_{2m}}{(\frac{\beta}{2}-m+1)_j(-\frac{\beta}{2}-m)_j},
$$

$$
\prod_{l=1}^{j-1}\big[(m-l+i)(m-l-i+1)\big] = (-m-i+1)_{j-1}(-m+i)_{j-1}
$$

allow to obtain

$$
R_m^{(1)}\big(i(i-1)\big) = (\tfrac{\beta}{2}-m+1)_{2m}(\tfrac{\beta}{2}+i)(\tfrac{\beta}{2}-i+1)\sum_{j=1}^{m}\frac{(-m-i+1)_{j-1}(-m+i)_{j-1}}{(\frac{\beta}{2}-m+1)_j(-\frac{\beta}{2}-m)_j}.
$$

Hence, our claim is equivalent to

$$\sum_{j=1}^{m} \frac{(-m-i+1)_{j-1}(-m+i)_{j-1}}{(\frac{\beta}{2}-m+1)_{j-1}(-\frac{\beta}{2}-m)_{j-1}} = -\frac{(\frac{\beta}{2}-m+1)(-\frac{\beta}{2}-m)}{(\frac{\beta}{2}+i)(\frac{\beta}{2}-i+1)}$$

for all $i = 1, \dots, m$. The identity

$$\frac{1}{(\frac{\beta}{2}+i)(\frac{\beta}{2}-i+1)}\left[\frac{(-m-i+1)_j(-m+i)_j}{(\frac{\beta}{2}-m+1)_{j-1}(-\frac{\beta}{2}-m)_{j-1}} - \frac{(-m-i+1)_{j-1}(-m+i)_{j-1}}{(\frac{\beta}{2}-m+1)_{j-2}(-\frac{\beta}{2}-m)_{j-2}}\right] = \frac{(-m-i+1)_{j-1}(-m+i)_{j-1}}{(\frac{\beta}{2}-m+1)_{j-1}(-\frac{\beta}{2}-m)_{j-1}}$$

implies that our sum is a telescoping sum, and the only term which survives the summation process is

$$\begin{aligned}\sum_{j=1}^{m} \frac{(-m-i+1)_{j-1}(-m+i)_{j-1}}{(\frac{\beta}{2}-m+1)_{j-1}(-\frac{\beta}{2}-m)_{j-1}} &= -\frac{1}{(\frac{\beta}{2}+i)(\frac{\beta}{2}-i+1)}\frac{1}{(\frac{\beta}{2}-m+2)_{-1}(-\frac{\beta}{2}-m+1)_{-1}}\\ &= -\frac{(\frac{\beta}{2}-m+1)(-\frac{\beta}{2}-m)}{(\frac{\beta}{2}+i)(\frac{\beta}{2}-i+1)},\end{aligned}$$

because $i = 1, \dots, m$. Finally, for the last evaluation point of our polynomials we take $\frac{\beta}{2}(\frac{\beta}{2}+1)$:

$$\begin{aligned}R_m(\tfrac{\beta}{2}(\tfrac{\beta}{2}+1);0) &= \prod_{l=1}^{m}(\tfrac{\beta}{2}(\tfrac{\beta}{2}+1) - l(l-1))\\ &= \prod_{l=1}^{m}\big((\tfrac{\beta}{2}+l)(\tfrac{\beta}{2}-l+1)\big) = (\tfrac{\beta}{2}-m+1)_{2m},\end{aligned}$$

which is exactly $R^{(1)}_m(\frac{\beta}{2}(\frac{\beta}{2}+1)) + (\frac{\beta}{2}-m+1)_{2m}$, since $R^{(1)}_m(\frac{\beta}{2}(\frac{\beta}{2}+1)) = 0$. This completes the proof. □

Proposition 2.5 *The set of polynomials $R^{(1)}_m$, $m \in \mathbb{N}_0$, satisfies the following recurrence relations*

$$R^{(1)}_{m+1}(y) = \big[y - m(m+1)\big]R^{(1)}_m(y) + (\tfrac{\beta}{2}-m+1)_{2m}\big[y - \tfrac{\beta}{2}(\tfrac{\beta}{2}+1)\big], \tag{2.8}$$

with $R^{(1)}_0(y) = 0$.

Proof. The proof is straightforward. Starting from

$$\begin{aligned}R^{(1)}_{m+1}(y) &= (\tfrac{\beta}{2}-m+1)_{2m}\big[y - \tfrac{\beta}{2}(\tfrac{\beta}{2}+1)\big]\\ &\quad + \sum_{j=2}^{m+1}(\tfrac{\beta}{2}-m+j)_{2m+2-2j} \times \big[y - \tfrac{\beta}{2}(\tfrac{\beta}{2}+1)\big]R_{j-1}(y;m+1),\end{aligned}$$

we substitute

$$R_{j-1}(y;m+1) = \big[y - m(m+1)\big]R_{j-2}(y;m)$$

and shift the summation index. This yields

$$R^{(1)}_{m+1}(y) = (\tfrac{\beta}{2} - m + 1)_{2m}\big[y - \tfrac{\beta}{2}(\tfrac{\beta}{2}+1)\big] + \big[y - m(m+1)\big]R^{(1)}_m(y),$$

which completes the proof. □

Finally, we introduce the set of polynomials

$$\begin{aligned} s^{(1)}_m(y) &:= \sum_{k=0}^{m} \binom{m}{k} (\lambda - \tfrac{\beta}{2} + k + 1)_{m-k}(\tfrac{\beta}{2} + k + 1)_{m-k-1}\times \\ &\qquad \times \big[k(\lambda+\beta+2m) + \tfrac{\beta}{2}(\lambda - \beta - 2m)\big]R^{(1)}_k(y) \\ &=: \sum_{k=0}^{m} D^{(1)}_k(m) R^{(1)}_k(y) \end{aligned} \tag{2.9}$$

of degree $m \in \mathbb{N}_0$, and we remark that $s^{(1)}_0(y) = 0$.

Proposition 2.6 *The set of polynomials $s^{(1)}_m$, $m \in \mathbb{N}_{\geq 2}$, satisfies*

$$\begin{aligned} s^{(1)}_m(y) = (\lambda - \beta + 2m)&s^{(-)}_m(y) \\ &- 2m(\lambda+2m)(\lambda - \tfrac{\beta}{2} + m)s^{(-)}_{m-1}(y) \\ &+ m(m-1)(\lambda+\beta+2m)(\lambda - \tfrac{\beta}{2} + m - 1)_2 s^{(-)}_{m-2}(y) \\ &- (\lambda)_m(\lambda-\beta)(\tfrac{\beta}{2})_m s^{(-)}_0(y). \end{aligned} \tag{2.10}$$

Notice that $s^{(-)}_0(y) = 1$.

Proof. By Proposition 2.4 and Lemma A.1, we have

$$\begin{aligned} s^{(1)}_m(y) &= \sum_{k=0}^{m} D^{(1)}_k(m) R_k(y;0) - \sum_{k=0}^{m} D^{(1)}_k(m)(\tfrac{\beta}{2} - k + 1)_{2k} \\ &= \sum_{k=0}^{m} D^{(1)}_k(m) R_k(y;0) - (\lambda)_m(\lambda-\beta)(\tfrac{\beta}{2})_m. \end{aligned}$$

Comparison of the coefficients in Equation (2.10) by $R_k(y;0)$ gives

$$\begin{aligned} D^{(1)}_m(m) &= (\lambda - \beta + 2m)C^{(-)}_m(m), \\ D^{(1)}_{m-1}(m) &= (\lambda - \beta + 2m)C^{(-)}_{m-1}(m) - 2m(\lambda+2m)(\lambda - \tfrac{\beta}{2} + m)C^{(-)}_{m-1}(m-1), \\ D^{(1)}_k(m) &= (\lambda - \beta + 2m)C^{(-)}_k(m) - 2m(\lambda+2m)(\lambda - \tfrac{\beta}{2} + m)C^{(-)}_k(m-1), \\ &\qquad + m(m-1)(\lambda+\beta+2m)(\lambda - \tfrac{\beta}{2} + m - 1)_2 C^{(-)}_k(m-2), \end{aligned}$$

for all $k \in \mathbb{N}_0$ such that $k \leq m-2$. Checking these identities is straightforward and the proof is complete. □

2.1. **Hypergeometric interpretation of combinatorial identities.** In this subsection we interpret our polynomials $s_m^{(\pm)}$ as the dual Hahn polynomials, cf. Appendix A. Furthermore, it follows from Proposition 2.6 that $s_m^{(1)}$ can be realized as a linear combination of the dual Hahn polynomials with y-independent coefficients. This linear combination can be rewritten as a sum of a hypergeometric polynomials of type $(4,3)$ and $(2,1)$ or, equivalently, as a linear combination with y-dependent coefficients of two dual Hahn polynomials.

Firstly, we will consider $R_k(\cdot;\alpha)$ in the variable $y(y+1)$,

$$R_k(y(y+1);\alpha) = (-1)^k(-y-\alpha)_k(y+1-\alpha)_k \tag{2.11}$$

for all $k \in \mathbb{N}$.

Secondly, we observe that by standard Pochhammer identities our polynomials $s_m^{(\pm)}$ are given by generalized hypergeometric functions of type $(3,2)$:

$$s_m^{(-)}\big(y(y+1)\big) = (\tfrac{\beta}{2}+2)_m(\lambda-\tfrac{\beta}{2}+1)_m \times {}_3F_2\left[\begin{matrix}-m\,,\ -y\,,\ 1+y\\ \frac{\beta}{2}+2\,,\ \lambda-\frac{\beta}{2}+1\end{matrix};1\right],$$

$$s_m^{(+)}\big(y(y+1)\big) = (\tfrac{\beta}{2})_m(\lambda-\tfrac{\beta}{2}+1)_m \times {}_3F_2\left[\begin{matrix}-m\,,\ -y\,,\ 1+y\\ \frac{\beta}{2}\,,\ \lambda-\frac{\beta}{2}+1\end{matrix};1\right],$$

and hence we can express them as the dual Hahn polynomials, see Appendix A,

$$s_m^{(-)}\big(y(y+1)\big) = (\tfrac{\beta}{2}+2)_m(\lambda-\tfrac{\beta}{2}+1)_m \times R_m\big(y(y+1);\tfrac{\beta}{2}+1,-1-\tfrac{\beta}{2},\tfrac{\beta}{2}-\lambda\big),$$
$$s_m^{(+)}\big(y(y+1)\big) = (\tfrac{\beta}{2})_m(\lambda-\tfrac{\beta}{2}+1)_m \times R_m\big(y(y+1);\tfrac{\beta}{2}-1,1-\tfrac{\beta}{2},\tfrac{\beta}{2}-\lambda\big).$$

When choosing $\lambda=\frac{\beta}{2}-N$ for some $N\in\mathbb{N}$, $\frac{\beta}{2}-\lambda=N$ becomes a positive integer, as required by definition of the dual Hahn polynomials.

Remark 2.7 One can easily realize that there is no hypergeometric series representative for $s_m^{(\pm)}(y)$. For example, the subleading coefficients in such an expansion do not factorize nicely into linear factors. Moreover, the quotients of successive coefficients are not rational functions in the summation index (which is, in fact, the defining property of a hypergeometric series).

Thirdly, as a consequence of Proposition 2.6, the polynomial $s_m^{(1)}$ is a linear combination of generalized hypergeometric functions of type $(3,2)$ and $(2,1)$,

$$\begin{aligned} s_m^{(1)}\big(y(y+1)\big) = (\lambda-\tfrac{\beta}{2}+1)_m\Big[&(\lambda-\beta+2m)(\tfrac{\beta}{2}+2)_m \times {}_3F_2\left[\begin{matrix}-m\,,\ -y\,,\ 1+y\\ \frac{\beta}{2}+2\,,\ \lambda-\frac{\beta}{2}+1\end{matrix};1\right]\\ &-2m(\lambda+2m)(\tfrac{\beta}{2}+2)_{m-1}\times {}_3F_2\left[\begin{matrix}-m+1\,,\ -y\,,\ 1+y\\ \frac{\beta}{2}+2\,,\ \lambda-\frac{\beta}{2}+1\end{matrix};1\right]\\ &+m(m-1)(\lambda+\beta+2m)(\tfrac{\beta}{2}+2)_{m-2}\times {}_3F_2\left[\begin{matrix}-m+2\,,\ -y\,,\ 1+y\\ \frac{\beta}{2}+2\,,\ \lambda-\frac{\beta}{2}+1\end{matrix};1\right]\\ &-(\lambda-\beta)(\tfrac{\beta}{2})_m\times {}_2F_1\left[\begin{matrix}-m\,,\ -\frac{\beta}{2}+1\\ \lambda-\frac{\beta}{2}+1\end{matrix};1\right]\Big]. \end{aligned} \tag{2.12}$$

Notice that the coefficients of the previous linear combination are y-independent.

We were informed by Christian Krattenthaler that our $s_m^{(1)}$ can be organized by the following two expressions based on various generalized hypergeometric functions:

Proposition 2.8 *The set of polynomials $s_m^{(1)}$, for $m \in \mathbb{N}$, has the following descriptions:*

$$s_m^{(1)}\big(y(y+1)\big) = (\lambda - \tfrac{\beta}{2}+1)_m(\tfrac{\beta}{2})_m\Big[(\lambda-\beta-2m)\times {}_4F_3\left[\begin{matrix}1+\gamma\,,-m\,,\ -y\,,\ 1+y\\ \gamma\,,\ \frac{\beta}{2}+1\,,\ \lambda-\frac{\beta}{2}+1\end{matrix};1\right] - (\lambda-\beta)\times {}_2F_1\left[\begin{matrix}-m\,,\ -\frac{\beta}{2}+1\\ \lambda-\frac{\beta}{2}+1\end{matrix};1\right]\Big], \tag{2.13}$$

where $\gamma := \frac{\beta(\lambda-\beta-2m)}{2(\lambda+\beta+2m)}$. Additionally it holds

$$s_m^{(1)}\big(y(y+1)\big) = (\lambda - \tfrac{\beta}{2}+1)_m(\tfrac{\beta}{2})_m(\lambda-\beta-2m)\times\Big[{}_3F_2\left[\begin{matrix}-m\,,\ -y\,,\ 1+y\\ \frac{\beta}{2}+1\,,\ \lambda-\frac{\beta}{2}+1\end{matrix};1\right] + \frac{2my(y+1)(\lambda+\beta+2m)}{\beta(\frac{\beta}{2}+1)(\lambda-\frac{\beta}{2}+1)(\lambda-\beta-2m)}\times {}_3F_2\left[\begin{matrix}1-m\,,\ 1-y\,,\ 2+y\\ \frac{\beta}{2}+2\,,\ \lambda-\frac{\beta}{2}+2\end{matrix};1\right]\Big] - (\lambda-\tfrac{\beta}{2}+1)_m(\tfrac{\beta}{2})_m(\lambda-\beta)\times {}_2F_1\left[\begin{matrix}-m\,,\ -\frac{\beta}{2}+1\\ \lambda-\frac{\beta}{2}+1\end{matrix};1\right], \tag{2.14}$$

where the coefficients in the linear combination (2.14) *are y-depended.*

Proof. The proof of Equation (2.13) is based on the elementary identity

$$\begin{aligned}&(\lambda-\beta+2m)(\tfrac{\beta}{2}+2)_m(-m)_k - 2m(\lambda+2m)(\tfrac{\beta}{2}+2)_{m-1}(1-m)_k\\ &\quad + m(m-1)(\lambda+\beta+2m)(\tfrac{\beta}{2}+2)_{m-2}(2-m)_k\\ &= \tfrac{1}{4}(\beta+2k+2)(-\beta^2+2\beta k+\lambda\beta+2\lambda k-2\beta m+4km)(\tfrac{\beta}{2}+2)_{m-2}(-m)_k\\ &= (\tfrac{\beta}{2})_m(\lambda-\beta-2m)\frac{(\gamma+1)_k(-m)_k}{(\gamma)_k(\frac{\beta}{2}+1)_k}.\end{aligned}$$

The standard Pochhammer identity $\frac{(\gamma+1)_k}{(\gamma)_k} = 1+\frac{k}{\gamma}$ allows to decompose our generalized hypergeometric function ${}_4F_3$ into two summands, which lead to Equation (2.14). The proof is complete. □

3. Boundary value problem for conformal infinity

We start with a brief reminder on the boundary value problem for conformal infinity and the Laplace operator acting on differential forms, see [AG08]. Then we proceed to its complete solution in the case when the conformal infinity contains the flat or an Einstein metric.

Let (M,h) be a Riemannian oriented manifold of dimension $n \geq 3$. Note that all statements given below extend to the semi-Riemannian setting by careful checking the number of appearances of minuses induced by the signature. The differential $\mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M)$ has a formal adjoint given by the codifferential $\delta^h = (-1)^{p+1}(\star^h)^{-1}\circ \mathrm{d}\circ\star^h$ when acting on $p+1$-forms. Here we denoted by $\star^h : \Omega^p(M)\to\Omega^{n-p}(M)$ the Hodge operator on (M,h). The form Laplacian

$$\Delta^h := \mathrm{d}\delta^h + \delta^h\mathrm{d} : \Omega^p(M) \to \Omega^p(M) \tag{3.1}$$

is formally self-adjoint differential operator of second order.

Consider the Poincaré-Einstein space (X, g_+) associated to (M, h), see Appendix B. A differential p-form ω on X uniquely decomposes (note a different convention compared to [AG08]) off the boundary as

$$\omega = \omega^{(+)} + \tfrac{\mathrm{d}r}{r} \wedge \omega^{(-)}, \tag{3.2}$$

for differential forms $\omega^{(+)} \in \Omega^p(X)$ and $\omega^{(-)} \in \Omega^{p-1}(X)$ characterized by trivial contraction with the normal vector field ∂_r. It is straightforward to verify that the form Laplacian on X acting in the splitting (3.2) is

$$\begin{aligned}\Delta^{g_+} &= \begin{pmatrix} -(r\partial_r)^2 + (n-2p)r\partial_r & 2\mathrm{d} \\ 0 & -(r\partial_r)^2 + (n+2-2p)r\partial_r \end{pmatrix} \\ &+ \begin{pmatrix} r^2\Delta^{h_r} - r(\star^{h_r})^{-1}[\partial_r, \star^{h_r}]r\partial_r & -r\big[\mathrm{d}, (\star^{h_r})^{-1}[\partial_r, \star^{h_r}]\big] \\ [r\partial_r, r^2\delta^{h_r}] & r^2\Delta^{h_r} - r\partial_r(\star^{h_r})^{-1}[r\partial_r, \star^{h_r}] \end{pmatrix} =: P + P'. \end{aligned} \tag{3.3}$$

Here $\star^{h_r}$ and δ^{h_r} denote the Hodge operator and codifferential with respect to the 1-parameter family of metrics h_r on M. In the case when n is odd, the form Laplacian can be expanded as a power series around $r = 0$, while in even dimensions n there appear additional $\log(r)$-terms coming from the Poincaré-Einstein metric, cf. [AG08, Lemma 2.1].

For $\lambda \in \mathbb{C}$, we consider the eigenequation

$$\Delta^{g_+}\omega = \lambda(n - 2p - \lambda)\omega \tag{3.4}$$

with ω a p-form on X. The boundary value problem for conformal infinity consists of finding an asymptotic solution $\omega \in \Omega^p(X)$ of Equation (3.4) with prescribed boundary value $\varphi \in \Omega^p(M)$. The construction of a solution for this boundary value problem is algorithmically described in [AG08]. For a manifold with general conformal structure (M, h), this algorithm is quite complicated due to the complexity in the construction of the Poincaré-Einstein metric. As we shall see in next subsections, there is rather explicit solution when the conformal infinity is metrizable by the flat or an Einstein metric.

3.1. **Conformally flat metric.** Let $(M, h) = (\mathbb{R}^n, \langle\cdot,\cdot\rangle_n)$ be the euclidean space. Then the associated Poincaré-Einstein metric can be realized as the hyperbolic metric

$$g_+ = x_{n+1}^{-2}(\mathrm{d}x_{n+1}^2 + h)$$

on the upper half space $\mathbb{R}^{n+1}_{>0}$. Consider the asymptotic expansion of a p-form on $\mathbb{R}^{n+1}$, given by

$$\omega = x_{n+1}^{\lambda}\big[\sum_{j\geq 0} x_{n+1}^j\omega_j^{(+)} + \tfrac{\mathrm{d}x_{n+1}}{x_{n+1}} \wedge \sum_{j\geq 0} x_{n+1}^j\omega_j^{(-)}\big] \tag{3.5}$$

for $\omega_j^{(+)} \in \Omega^p(\mathbb{R}^n)$ and $\omega_j^{(-)} \in \Omega^{p-1}(\mathbb{R}^n)$. Formally, one can solve Equation (3.4) for a given initial data $\omega_0^{(+)} = \varphi \in \Omega^p(\mathbb{R}^n)$ in terms of the solution operators

$$\begin{aligned} \mathcal{T}_{2j}^{(+)}(\lambda) &: \Omega^p(\mathbb{R}^n) \to \Omega^p(\mathbb{R}^n), \\ \mathcal{T}_{2j}^{(-)}(\lambda) &: \Omega^{p-1}(\mathbb{R}^n) \to \Omega^{p-1}(\mathbb{R}^n), \end{aligned} \tag{3.6}$$

which are h-natural differential operators with rational polynomial coefficients in λ determining $\omega_{2j}^{(+)} = \mathcal{T}_{2j}^{(+)}(\lambda)\varphi$ and $\omega_{2j}^{(-)} = \mathcal{T}_{2j}^{(-)}(\lambda)\varphi$ uniquely for all $j \in \mathbb{N}_0$. Notice that

the solution operators turn out to be well-defined for $\lambda \neq \frac{n}{2} - p - j$ and $\lambda \neq n - 2p$, and by construction $\omega^{(\pm)}_{2j-1} = 0$ for $j \in \mathbb{N}$.

Remark 3.1 Due to the absence of curvature, the solution operators $\mathcal{T}^{(\pm)}_{2j}(\lambda)$ are given in terms of $\delta^h(\mathrm{d}\delta^h)^{j-1}$, $(\delta^h\mathrm{d})^j$ and $(\mathrm{d}\delta^h)^j$.

Proposition 3.2 *Let (M,h) be the euclidean space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_n)$, and let $j \in \mathbb{N}$ and $\lambda \neq \frac{n}{2} - p - N$ for all $N \in \{1,\dots,j\}$. Then*

$$\mathcal{T}^{(-)}_{2j}(\lambda) = \frac{1}{4^{j-1}(j-1)!(\lambda-\frac{n-2p}{2}+1)_{j-1}(\lambda-n+2p)}\delta^h(\mathrm{d}\delta^h)^{j-1},$$

$$\mathcal{T}^{(+)}_{2j}(\lambda) = \frac{1}{4^j j!(\lambda-\frac{n-2p}{2}+1)_j(\lambda-n+2p)}\big[(\lambda-n+2p)(\delta^h\mathrm{d})^j + (\lambda-n+2p+2j)(\mathrm{d}\delta^h)^j\big] \quad (3.7)$$

for $\mathcal{T}^{(-)}_0(\lambda) = 0$, $\mathcal{T}^{(+)}_0(\lambda) = \mathrm{Id}$.

Proof. For $(M,h) = (\mathbb{R}^n, \langle\cdot,\cdot\rangle_n)$ and $r = x_{n+1}$, P', see Equation (3.3), reduces to

$$P' = x^2_{n+1}\begin{pmatrix}\Delta^h & 0\\ 2\delta^h & \Delta^h\end{pmatrix},$$

where all operators are considered with respect to $h = \langle\cdot,\cdot\rangle_n$. The ansatz (3.5) solves Equation (3.4) iff the following system is satisfied:

$$\big[(2j-2)(2\lambda-n+2p+2j-2) + 2(\lambda-n+2p+2j-2)\big]\omega^{(-)}_{2j} = \Delta^h\omega^{(-)}_{2j-2} + 2\delta^h\omega^{(+)}_{2j-2},$$
$$2j(2\lambda-n+2p+2j)\omega^{(+)}_{2j} = \Delta^h\omega^{(+)}_{2j-2} + 2\mathrm{d}\omega^{(-)}_{2j}$$

for $j \in \mathbb{N}$ and $\omega^{(+)}_0 \in \Omega^p(\mathbb{R}^n)$ arbitrary, while $\omega^{(-)}_0 = 0$. It is now straightforward to check that the solution operators satisfy the recurrence relations and the proof is complete. □

3.2. **Conformally Einstein metrics.** Let (M,h) be an Einstein manifold normalized by $Ric(h) = 2\lambda(n-1)h$ for some constant $\lambda \in \mathbb{R}$. This implies that the (normalized) scalar curvature and the Schouten tensor are given by $J = n\lambda$ and $P = \frac{J}{n}h$, respectively. In this case the Poincaré-Einstein metric is of the form

$$g_+ = r^{-2}\big(\mathrm{d}r^2 + J(r)^2h\big), \quad (3.8)$$

for $J(r) := (1 - \frac{J}{2n}r^2)$. The polynomial type of $J(r)$ implies that one can explicitly compute the form Laplacian, especially the term P' in Equation (3.3). From now on we use the abbreviation $\beta := n - 2p$.

Lemma 3.3 *Let (M,h) be an Einstein manifold, normalized by $Ric(h) = 2\lambda(n-1)h$. Then in the splitting* (3.2)*, it holds*

$$\Delta^{g_+} = \begin{pmatrix}-(r\partial_r)^2 + \beta r\partial_r & 0\\ 0 & -(r\partial_r)^2 + (\beta+2)r\partial_r\end{pmatrix} + \begin{pmatrix}A_1 & A_2\\ A_3 & A_4\end{pmatrix}, \quad (3.9)$$

where

$$A_1 = r^2J(r)^{-2}\Delta^h + \beta\tfrac{J}{n}r^2J(r)^{-1}r\partial_r,$$

$$\begin{aligned}
A_2 =&2(1+\tfrac{J}{2n}r^2)J(r)^{-1}d,\\
A_3 =&2r^2(1+\tfrac{J}{2n}r^2)J(r)^{-3}\delta^h,\\
A_4 =&r^2J(r)^{-2}\Delta^h+(\beta+2)\tfrac{J}{n}r^2J(r)^{-2}(2+J(r)r\partial_r).
\end{aligned}$$

Proof. The explict formula for h_r, cf. (3.8), leads to the explicit form of Equation (3.3). In more detail, we note

$$\star^{h_r}(\eta)=J(r)^{\beta}\star^h(\eta),$$

for $\eta\in\Omega^p(M)$. Hence, we get on p-forms

$$\begin{aligned}
\delta^{h_r}&=(-1)^p(\star^{h_r}_{p-1})^{-1}\circ d\circ\star^{h_r}_p\\
&=(-1)^pJ(r)^{2(p-1)-n}J(r)^{n-2p}(\star^h_{p-1})^{-1}\circ d\circ\star^h_p=J(r)^{-2}\delta^h,
\end{aligned}$$

which implies $\Delta^{h_r}=J(r)^{-2}\Delta^h$. Furthermore, for p and $(p-1)$-forms $\omega^{(+)},\omega^{(-)}$ as introduced in (3.2), we have

$$\begin{aligned}
r(\star^{h_r})^{-1}[\partial_r,\star^{h_r}]r\partial_r\omega^{(+)}&=-\beta\tfrac{J}{n}J(r)^{-1}r^3\partial_r\omega^{(+)},\\
r\big[d,(\star^{h_r})^{-1}[\partial_r,\star^{h_r}]\big]\omega^{(-)}&=-2\tfrac{J}{n}r^2J(r)^{-1}d\omega^{(-)},\\
[r\partial_r,r^2\delta^{h_r}]\omega^{(+)}&=2r^2(1+\tfrac{J}{2n}r^2)J(r)^{-3}\delta^h\omega^{(+)},\\
r\partial_r(\star^{h_r})^{-1}[r\partial_r,\star^{h_r}]\omega^{(-)}&=-(\beta+2)r^2\tfrac{J}{n}J(r)^{-2}(2+J(r)r\partial_r)\omega^{(-)}.
\end{aligned}$$

The result then follows from (3.3). ☐

The eigenequation (3.4), acting in the splitting $\omega\simeq\begin{pmatrix}\omega^{(+)}\\ \omega^{(-)}\end{pmatrix}$, is equivalent to the system

$$\begin{aligned}
&\lambda(\beta-\lambda)\omega^{(-)}+(r\partial_r)^2\omega^{(-)}-(\beta+2)r\partial_r\omega^{(-)}\\
&\quad=2r^2(1+\tfrac{J}{2n}r^2)J(r)^{-3}\delta^h\omega^{(+)}+r^2J(r)^{-2}\Delta^h\omega^{(-)}\\
&\qquad+(\beta+2)\tfrac{2J}{n}r^2J(r)^{-2}\omega^{(-)}+(\beta+2)\tfrac{J}{n}r^2J(r)^{-1}r\partial_r\omega^{(-)}, \qquad (3.10)\\
&\lambda(\beta-\lambda)\omega^{(+)}+(r\partial_r)^2\omega^{(+)}-\beta r\partial_r\omega^{(+)}\\
&\quad=2(1+\tfrac{J}{2n}r^2)J(r)^{-1}d\omega^{(-)}+r^2J(r)^{-2}\Delta^h\omega^{(+)}+\beta\tfrac{J}{n}r^2J(r)^{-1}r\partial_r\omega^{(+)}. \qquad (3.11)
\end{aligned}$$

Using the polynomial type of $J(r)=1-\frac{J}{2n}r^2$, we multiply Equation (3.10) by $J(r)^3$ and Equation (3.11) by $J(r)^2$. As a result, the coefficients in both equations are polynomials of degree 3 and 2 in r^2, respectively. This is the key step to formulate

Proposition 3.4 *Let (M,h) be an Einstein manifold with the normalization given by $Ric(h)=\frac{2J}{n}(n-1)h$ for a constant $J\in\mathbb{R}$. The eigenequation (3.4) acting on*

$$\omega=\sum_{j\geq0}r^{\lambda+j}\omega_j^{(+)}+\tfrac{dr}{r}\wedge\sum_{j\geq0}r^{\lambda+j}\omega_j^{(-)},$$

for some (unknown) differential forms $\omega_j^{(+)} \in \Omega^p(M)$ and $\omega_j^{(-)} \in \Omega^{p-1}(M)$, is equivalent to the following recurrence relations:

$$A_j w_j^{(-)} = B_j(\tfrac{J}{2n}) w_{j-2}^{(-)} + C_j(\tfrac{J}{2n})^2 w_{j-4}^{(-)} + D_j(\tfrac{J}{2n})^3 w_{j-6}^{(-)} + \Delta^h w_{j-2}^{(-)} - \tfrac{J}{2n}\Delta^h \omega_{j-4}^{(-)} + 2\delta^h w_{j-2}^{(+)} + 2(\tfrac{J}{2n})\delta^h w_{j-4}^{(+)}, \tag{3.12}$$

$$a_j w_j^{(+)} = b_j(\tfrac{J}{2n}) w_{j-2}^{(+)} + c_j(\tfrac{J}{2n})^2 w_{j-4}^{(+)} + \Delta^h w_{j-2}^{(+)} + 2dw_j^{(-)} - 2(\tfrac{J}{2n})^2 dw_{j-4}^{(-)}, \tag{3.13}$$

with coefficients

$$\begin{aligned}
A_j &:= -\big[(j-2)(\beta-2\lambda-j+2) + 2(\beta-\lambda-j+2)\big],\\
B_j &:= -3\big[(j-4)(\beta-2\lambda-j+4) + 2(\beta-\lambda-j+4)\big] + 2(\beta+2)(\lambda+j),\\
C_j &:= 3\big[(j-6)(\beta-2\lambda-j+6) + 2(\beta-\lambda-j+6)\big] - 4(\beta+2)(\lambda+j-3),\\
D_j &:= -\big[(j-8)(\beta-2\lambda-j+8) + 2(\beta-\lambda-j+8)\big] + 2(\beta+2)(\lambda+j-6),\\
a_j &:= -j(\beta-2\lambda-j),\\
b_j &:= 2\big[(j-2)(\lambda+j-2) + \lambda(\beta+j-2)\big],\\
c_j &:= -(2\lambda+j-4)(\beta+j-4),
\end{aligned} \tag{3.14}$$

depending on $j \in \mathbb{N}$, $\beta = n-2p$, $\lambda \in \mathbb{C}$, and the initial data $\omega_0^{(-)} = 0$ and $\omega_0^{(+)} \in \Omega^{(p)}(M)$. Furthermore, it holds $\omega_{2j-1}^{(\pm)} = 0$ for all $j \in \mathbb{N}$.

Proof. Due to the polynomial type of the coefficients in Equations (3.10) and (3.11) after multiplication with appropriate powers of $J(r)$, and the ansatz for ω, we obtain the recurrence relations by comparing the coefficients by $r^{\lambda+j}$ for $j \in \mathbb{N}_0$. Based on $\omega_0^{(-)} = 0$ and evenness of involved coefficients in r, we get $\omega_{2j-1}^{(\pm)} = 0$ for all $j \in \mathbb{N}$. This completes the proof. □

In order to get an insight into the solution structure of the recurrence relations (3.12) and (3.13), we present several low-order approximations:

Second-order approximation: The relation (3.12) for $j=2$ gives

$$2(\lambda-\beta)\omega_2^{(-)} = 2\delta^h \omega_0^{(+)},$$

and (3.13) for $j=2$ implies

$$4(2\lambda-\beta+2)(\lambda-\beta)\omega_2^{(+)} = 2(\lambda-\beta)R_1^{(1)}\omega_0^{(+)} + 2(\lambda-\beta+2)R_1^{(2)}\omega_0^{(+)} + 2(2\lambda-\beta+2)(\lambda-\beta)\beta\tfrac{J}{2n}\omega_0^{(+)}$$

for

$$R_1^{(1)} := \delta^h d + \tfrac{\beta}{2}(\tfrac{\beta}{2}-1)\tfrac{2J}{n}, \quad R_1^{(2)} := d\delta^h.$$

Hence $\omega_2^{(-)}$ and $\omega_2^{(+)}$ are well-defined for $\lambda \neq \beta$ and $\lambda \neq \frac{\beta}{2}-1, \beta$, respectively. As we will see later, for $\lambda = \frac{\beta}{2}-1$ the right hand side of Equation (**??**) is proportional to the second-order Branson-Gover operator.

Fourth-order approximation: The relation (3.12) for $j=4$ gives

$$4(2\lambda-\beta+2)(\lambda-\beta)\omega_4^{(-)} = 2R_1^{(0)}\delta^h\omega_0^{(+)} + 2(2\lambda-\beta+2)(\beta+4)(\tfrac{J}{2n})\delta^h\omega_0^{(+)}$$

in terms of the operator

$$R_1^{(0)} := \delta^h d + \tfrac{\beta}{2}(\tfrac{\beta}{2}+1)\tfrac{2J}{n}.$$

The relation (3.13) for $j=4$ reduces to

$$\begin{aligned}
&16(2\lambda-\beta+4)(2\lambda-\beta+2)(\lambda-\beta)\omega_4^{(+)}\\
&\qquad = 2(\lambda-\beta)R_2^{(1)}\omega_0^{(+)} + 2(\lambda-\beta+4)R_2^{(2)}\omega_0^{(+)}\\
&\qquad\quad + 4(2\lambda-\beta+4)(\lambda-\beta)(\beta+2)(\tfrac{J}{2n})R_1^{(1)}\omega_0^{(+)}\\
&\qquad\quad + 4(2\lambda-\beta+4)\big[2(\lambda-\beta+4)+\beta(\lambda-\beta)\big](\tfrac{J}{2n})R_1^{(2)}\omega_0^{(+)}\\
&\qquad\quad + 2(2\lambda-\beta+4)(2\lambda-\beta+2)(\lambda-\beta)\beta(\beta+2)(\tfrac{J}{2n})^2\omega_0^{(+)},
\end{aligned}$$

in terms of additional operators

$$\begin{aligned}
R_2^{(1)} &:= \big[\delta^h d + (\tfrac{\beta}{2}-1)\tfrac{\beta}{2}\tfrac{2J}{n}\big]\big[\delta^h d + (\tfrac{\beta}{2}-2)(\tfrac{\beta}{2}+1)\tfrac{2J}{n}\big],\\
R_2^{(2)} &:= \tfrac{\beta}{2}(\tfrac{\beta}{2}+1)(\tfrac{2J}{n})d\delta^h + d\delta^h\big[d\delta^h + (\tfrac{\beta}{2}-1)(\tfrac{\beta}{2}+2)\tfrac{2J}{n}\big].
\end{aligned}$$

Hence $\omega_4^{(-)}$ and $\omega_4^{(+)}$ are well-defined for $\lambda \neq \frac{\beta}{2}-1, \beta$ and $\lambda \neq \frac{\beta}{2}-1, \frac{\beta}{2}-2, \beta$, respectively. For $\lambda = \frac{\beta}{2}-2$ the right hand side of Equation (**??**) is proportional to the fourth-order Branson-Gover operator.

Sixth-order approximation: The relation (3.12) for $j=6$ gives

$$\begin{aligned}
&16(2\lambda-\beta+2)(2\lambda-\beta+4)(\lambda-\beta)\omega_6^{(-)}\\
&\qquad = 2R_2^{(0)}\delta^h\omega_0^{(+)} + 4(2\lambda-\beta+4)(\beta+6)(\tfrac{J}{2n})R_1^{(0)}\delta^h\omega_0^{(+)}\\
&\qquad\quad + 2(2\lambda-\beta+2)(2\lambda-\beta+4)(\beta+4)(\beta+6)(\tfrac{J}{2n})^2\delta^h\omega_0^{(+)}
\end{aligned}$$

in terms of the additional operator

$$R_2^{(0)} := \big[\delta^h d + \tfrac{\beta}{2}(\tfrac{\beta}{2}+1)\tfrac{2J}{n}\big]\big[\delta^h d + (\tfrac{\beta}{2}-1)(\tfrac{\beta}{2}+2)\tfrac{2J}{n}\big].$$

The relation (3.13) for $j=6$ yields after some computations

$$\begin{aligned}
&96(2\lambda-\beta+6)(2\lambda-\beta+4)(2\lambda-\beta+2)(\lambda-\beta)\omega_6^{(+)}\\
&\ = 2(\lambda-\beta)R_3^{(1)}\omega_0^{(+)} + 2(\lambda-\beta+6)R_3^{(2)}\omega_0^{(+)}\\
&\quad + 6(2\lambda-\beta+6)(\lambda-\beta)(\beta+4)(\tfrac{J}{2n})R_2^{(1)}\omega_0^{(+)}\\
&\quad + 6(2\lambda-\beta+6)\big[4(\lambda-\beta+6)+\beta(\lambda-\beta+2)\big](\tfrac{J}{2n})R_2^{(2)}\omega_0^{(+)}\\
&\quad + 6(2\lambda-\beta+6)(2\lambda-\beta+4)(\lambda-\beta)(\beta+2)(\beta+4)(\tfrac{J}{2n})^2R_1^{(1)}\omega_0^{(+)}\\
&\quad + 6(2\lambda-\beta+6)(2\lambda-\beta+4)\big[2(\lambda-\beta+6)+\beta(\lambda-\beta-2)\big](\beta+4)(\tfrac{J}{2n})^2R_1^{(2)}\omega_0^{(+)}\\
&\quad + 2(2\lambda-\beta+6)(2\lambda-\beta+4)(2\lambda-\beta+2)(\lambda-\beta)\beta(\beta+2)(\beta+4)(\tfrac{J}{2n})^3\omega_0^{(+)}
\end{aligned}$$

expressed in terms of additional operators

$$\begin{aligned}
R_3^{(1)} &:= \big[\delta^h d + (\tfrac{\beta}{2}-1)\tfrac{\beta}{2}\tfrac{2J}{n}\big]\big[\delta^h d + (\tfrac{\beta}{2}-2)(\tfrac{\beta}{2}+1)\tfrac{2J}{n}\big]\big[\delta^h d + (\tfrac{\beta}{2}-3)(\tfrac{\beta}{2}+2)\tfrac{2J}{n}\big],\\
R_3^{(2)} &:= (\tfrac{\beta}{2}-1)\tfrac{\beta}{2}(\tfrac{\beta}{2}+1)(\tfrac{\beta}{2}+2)(\tfrac{2J}{n})^2d\delta^h + \tfrac{\beta}{2}(\tfrac{\beta}{2}+1)\tfrac{2J}{n}d\delta^h\big[d\delta^h + (\tfrac{\beta}{2}-2)(\tfrac{\beta}{2}+3)\tfrac{2J}{n}\big]\\
&\quad + d\delta^h\big[d\delta^h + (\tfrac{\beta}{2}-1)(\tfrac{\beta}{2}+2)\tfrac{2J}{n}\big]\big[d\delta^h + (\tfrac{\beta}{2}-2)(\tfrac{\beta}{2}+3)\tfrac{2J}{n}\big].
\end{aligned}$$

Hence $\omega_6^{(-)}$ and $\omega_6^{(+)}$ are well-defined for $\lambda \neq \frac{\beta}{2}-1, \frac{\beta}{2}-2, \beta$ and $\lambda \neq \frac{\beta}{2}-1, \frac{\beta}{2}-2, \frac{\beta}{2}-3, \beta$, respectively. For $\lambda = \frac{\beta}{2}-3$ the right hand side of Equation (**??**) is proportional to the sixth-order Branson-Gover operator.

The previous approximations indicate the following definition of the solution operators:

$$\begin{aligned}
\mathcal{T}_m^{(-)}(\lambda) :=& \big[(\lambda-\beta)\prod_{k=1}^{m-1} a_{2k}\big]^{-1} (\tfrac{2J}{n})^{m-1} s_{m-1}^{(-)}(y^{(-)}) \circ \delta^h, \\
\mathcal{T}_m^{(+)}(\lambda) :=& \big[(\lambda-\beta)\prod_{k=1}^{m} a_{2k}\big]^{-1} (\tfrac{2J}{n})^{m} \Big[(\lambda-\beta) s_m^{(+)}(y^{(+)}) + s_m^{(1)}(y^{(1)})\Big],
\end{aligned} \tag{3.15}$$

for $m \in \mathbb{N}$. Here we have taken the evaluation of the polynomials $s_k^{(\pm)}$ and $s_k^{(1)}$ at

$$\begin{aligned}
y^{(-)} &:= \tfrac{n}{2J}\delta^h \mathrm{d} + \tfrac{\beta}{2}(\tfrac{\beta}{2}+1), \\
y^{(+)} &:= \tfrac{n}{2J}\delta^h \mathrm{d} + \tfrac{\beta}{2}(\tfrac{\beta}{2}-1), \\
y^{(1)} &:= \tfrac{n}{2J}\mathrm{d}\delta^h + \tfrac{\beta}{2}(\tfrac{\beta}{2}+1).
\end{aligned}$$

Remark 3.5 The inspiration for the definition of $\mathcal{T}_m^{(+)}(\lambda)$ comes from the scalar case, cf. [FG11, Chapter 7]. Restricting to 0-forms, Equation (3.12) becomes trivial and $\mathcal{T}_m^{(-)}(\lambda) = 0$, while Equation (3.13) is solved by $\mathcal{T}_m^{(+)}(\lambda)$ with vanishing term $s_m^{(1)}$.

The proof of the next theorem is mainly based on the combinatorial identities discussed in Section 2.

Theorem 3.6 *Let $m \in \mathbb{N}$, $\varphi \in \Omega^p(M)$ and $\lambda \neq \frac{\beta}{2} - N$ for all $N \in \{1, \ldots, m\}$. Then the solution of the recurrence relation* (3.12) *and* (3.13) *is given by*

$$\begin{aligned}
\omega_{2m}^{(-)} =& \mathcal{T}_m^{(-)}(\lambda)\omega_0^{(+)}, \\
\omega_{2m}^{(+)} =& \mathcal{T}_m^{(+)}(\lambda)\omega_0^{(+)},
\end{aligned} \tag{3.16}$$

with the boundary data $\omega_0^{(-)} = 0$, $\omega_0^{(+)} = \varphi$.

If $\lambda = \frac{\beta}{2} - N$ for some $N \in \{1, \ldots, m\}$, this solution holds for all $m < N$.

Proof. In order to shorten the notation, we introduce

$$B_m^{(+)} := \big[(\lambda-\beta)\prod_{k=1}^{m} a_{2k}\big]^{-1}.$$

First of all, we verify the recurrence (3.13). In terms of solution operators, it reads

$$\begin{aligned}
a_{2m}\mathcal{T}_m^{(+)}(\lambda) = b_{2m}(\tfrac{J}{2n})\mathcal{T}_{m-1}^{(+)}(\lambda) + c_{2m}(\tfrac{J}{2n})^2\mathcal{T}_{m-2}^{(+)}(\lambda) + \Delta^h\mathcal{T}_{m-1}^{(+)}(\lambda) \\
+ 2\mathrm{d}\mathcal{T}_m^{(-)}(\lambda) - 2(\tfrac{J}{2n})^2\mathrm{d}\mathcal{T}_{m-2}^{(-)}(\lambda).
\end{aligned}$$

This can be decomposed, due to $\mathrm{d}^2 = 0 = (\delta^h)^2$ and dealing with $\mathrm{d}\delta^h$ and $\delta^h\mathrm{d}$ as independent commuting variables, into two independent claims. The first is

$$a_{2m}B^{(+)}_m s^{(+)}_m(y^{(+)}) = B^{(+)}_{m-1}\big[\tfrac{n}{2J}\delta^h\mathrm{d} + \tfrac{1}{4}b_{2m}\big]s^{(+)}_{m-1}(y^{(+)}) + \tfrac{1}{16}c_{2m}B^{(+)}_{m-2}s^{(+)}_{m-2}(y^{(+)}), \quad (3.17)$$

while the second is given by

$$\begin{aligned} a_{2m}B^{(+)}_m s^{(1)}_m(y^{(1)}) &= B^{(+)}_{m-1}\big[\tfrac{n}{2J}\mathrm{d}\delta^h + \tfrac{1}{4}b_{2m}\big]s^{(1)}_{m-1}(y^{(1)}) + \tfrac{1}{16}c_{2m}B^{(+)}_{m-2}s^{(1)}_{m-2}(y^{(1)}) \\ &\quad + B^{(+)}_{m-1}(\lambda-\beta)\tfrac{n}{2J}\mathrm{d}\delta^h s^{(+)}_{m-1}(y^{(+)}) \\ &\quad + 2B^{(+)}_{m-1}\tfrac{n}{2J}\mathrm{d}s^{(-)}_m(y^{(-)})\delta^h - \tfrac{1}{8}B^{(+)}_{m-3}\tfrac{n}{2J}\mathrm{d}s^{(-)}_{m-2}(y^{(-)})\delta^h. \end{aligned} \quad (3.18)$$

Note that

$$\begin{aligned} a_{2k}B^{(+)}_k &= B^{(+)}_{k-1}, \\ \tfrac{1}{4}b_{2m} &= 2(m-1)(\lambda+m-1) + \tfrac{\beta}{2}\lambda, \\ \tfrac{1}{16}c_{2m}a_{2m-2} &= -(m-1)(\lambda+m-2)(\lambda-\tfrac{\beta}{2}+m-1)(\tfrac{\beta}{2}+m-2). \end{aligned}$$

We first notice that Equation (3.17) was proved in Proposition 2.3. Now we proceed with Equation (3.18). By $(\delta^h)^2 = 0$ and Lemma A.1, we have

$$\mathrm{d}\delta^h s^{(+)}_k(y^{(+)}) = s^{(+)}_k\big(\tfrac{\beta}{2}(\tfrac{\beta}{2}-1)\big)\mathrm{d}\delta^h = (\lambda)_k(\tfrac{\beta}{2})_k\mathrm{d}\delta^h.$$

Furthermore,

$$\mathrm{d}s^{(-)}_k(y^{(-)})\delta^h = \mathrm{d}\delta^h s^{(-)}_k(y^{(1)})$$

for all $k \in \mathbb{N}$. Applying Proposition 2.6 to $s^{(1)}_k(y^{(1)})$, for $k = m, m-1, m-2$, allows to rewrite Equation (3.18) just in terms of $s^{(-)}_k(y^{(1)})$ for appropriate collection of values of k. It turns out that Equation (3.18) is equivalent to the three-times repeated application of the recurrence relation in Proposition 2.2 to

$$\begin{aligned} &(\lambda-\beta+2m)s^{(-)}_{m+1}(y^{(1)}) - 2(m-1)(\lambda+2m-2)(\lambda-\tfrac{\beta}{2}+m-1)s^{(-)}_m(y^{(1)}) \\ &\qquad + (m-1)(m-2)(\lambda-\beta+2m-4)(\lambda-\tfrac{\beta}{2}+m-2)_2 s^{(-)}_{m-1}(y^{(1)}), \end{aligned}$$

which finally proves Equation (3.18).

Now we proceed to prove

$$\begin{aligned} A_{2m}\mathcal{T}^{(-)}_m(\lambda) &= B_{2m}(\tfrac{J}{2n})\mathcal{T}^{(-)}_{m-1}(\lambda) + C_{2m}(\tfrac{J}{2n})^2\mathcal{T}^{(-)}_{m-2}(\lambda) + D_{2m}(\tfrac{J}{2n})^3\mathcal{T}^{(-)}_{m-3}(\lambda) \\ &\quad + \Delta^h\mathcal{T}^{(-)}_{m-1}(\lambda) - (\tfrac{J}{2n})\Delta^h\mathcal{T}^{(-)}_{m-2}(\lambda) + 2\delta^h\mathcal{T}^{(+)}_{m-1}(\lambda) + 2(\tfrac{J}{2n})\delta^h\mathcal{T}^{(+)}_{m-2}(\lambda). \end{aligned} \quad (3.19)$$

Using two ingredients: due to $(\delta^h)^2 = 0$, we have

$$\begin{aligned} \Delta^h\mathcal{T}^{(-)}_k(\lambda) &= \delta^h d\mathcal{T}^{(-)}_k(\lambda), \\ \delta^h s^{(1)}_k(y^{(1)}) &= s^{(1)}_k(y^{(-)})\delta^h, \end{aligned}$$

and due to Lemma A.1., we get

$$\delta^h s^{(+)}_k(y^{(+)}) = \sum_{j=0}^{k} C^{(+)}_j(k)(\tfrac{\beta}{2}-j)_{2j}\delta^h = (\lambda)_k(\tfrac{\beta}{2})_k\delta^h,$$

we see that Equation (3.19) is equivalent to

$$\begin{aligned} A_{2m}B^{(+)}_{m-1}s^{(-)}_m(y^{(-)}) &= B^{(+)}_{m-2}\big[\tfrac{n}{2J}\delta^h d + \tfrac{1}{4}B_{2m}\big]s^{(-)}_{m-1}(y^{(-)}) \\ &\quad - \tfrac{1}{4}B^{(+)}_{m-3}\big[\tfrac{n}{2J}\delta^h d - \tfrac{1}{4}C_{2m}\big]s^{(-)}_{m-2}(y^{(-)}) \\ &\quad + \tfrac{1}{64}B^{(+)}_{m-4}D_{2m}s^{(-)}_{m-3}(y^{(-)}) \\ &\quad + 2B^{(+)}_{m-1}s^{(1)}_{m-1}(y^{(-)}) + 2B^{(+)}_{m-1}(\lambda)_{m-1}(\lambda-\beta)(\tfrac{\beta}{2})_{m-1} \\ &\quad + \tfrac{1}{2}B^{(+)}_{m-2}s^{(1)}_{m-2}(y^{(-)}) + \tfrac{1}{2}B^{(+)}_{m-2}(\lambda)_{m-2}(\lambda-\beta)(\tfrac{\beta}{2})_{m-2}. \end{aligned} \tag{3.20}$$

We replace the terms $s^{(1)}_{m-1}(y^{(-)})$ and $s^{(1)}_{m-2}(y^{(-)})$, using Proposition 2.6, by $s^{(-)}_k(y^{(-)})$ for appropriate collection of values of k. Then it turns out that Equation (3.20) is equivalent to the two-times application of the recurrence relation in Proposition 2.2 to

$$s^{(-)}_m(y^{(-)}) - a_{2m-4}s^{(-)}_{m-1}(y^{(-)}).$$

This proves the theorem. □

4. Applications: Branson-Gover operators on Einstein manifolds

This section is focused on the origin and properties of the Branson-Gover operators and their derived quantities on Einstein manifolds. In addition, we present another proof of a result in [GŠ13] on the decomposition of Branson-Gover operators as a product of second-order differential operators.

Let (M,h) be a Riemannian manifold of dimension n. For $p=0,\dots,n$ the Branson-Gover operators [BG05] are differential operators

$$L^{(p)}_{2N}:\Omega^p(M)\to\Omega^p(M)$$

of order $2N$, for $N\in\mathbb{N}$ ($N\le\frac{n}{2}$ for even n), of the form

$$L^{(p)}_{2N} = (\tfrac{n-2p}{2}+N)(\delta^h\mathrm{d})^N + (\tfrac{n-2p}{2}-N)(\mathrm{d}\delta^h)^N + \mathrm{LOT},$$

where LOT is the shorthand notation for the lower order (curvature correction) terms. They generalize the GJMS operator [GJMS92]

$$P_{2N} = (\Delta^h)^N + \mathrm{LOT}:\mathcal{C}^\infty(M)\to\mathcal{C}^\infty(M)$$

in the sense that $L^{(0)}_{2N} = (\frac{n}{2}+N)P_{2N}$. The key property of Branson-Gover operators is that they are conformally covariant,

$$e^{(\frac{n}{2}-p+N)\sigma}\circ\hat{L}^{(p)}_{2N} = L^{(p)}_{2N}\circ e^{(\frac{n}{2}-p-N)\sigma},\quad \sigma\in\mathcal{C}^\infty(M).$$

Here $\hat{\cdot}$ denotes the evaluation with respect to the conformally related metric $\hat{h}=e^{2\sigma}h$. In the case of even dimensions n and $p\le\frac{n}{2}$, the critical Branson-Gover operators factorize

$$\begin{aligned} L^{(p)}_{n-2p} &= G^{(p+1)}_{n-2p-1}\circ\mathrm{d} \\ &= \delta^h\circ Q^{(p+1)}_{n-2p-2}\circ\mathrm{d} \end{aligned}$$

by two additional differential operators

$$G^{(p+1)}_{n-2p-1} = (n-2p)\delta^h\circ\big((\mathrm{d}\delta)^{\frac{n}{2}-p-1}+\mathrm{LOT}\big):\Omega^{p+1}(M)\to\Omega^p(M),$$

$$Q^{(p+1)}_{n-2p-2} = (n-2p)\big((\mathrm{d}\delta^h)^{\frac{n}{2}-p-1} + \mathrm{LOT}\big) : \Omega^{p+1}(M) \to \Omega^{p+1}(M),$$

called the gauge companion and the Q-curvature operator, respectively. Similarly to $L^{(p)}_{2N}$, these relatives are quite complicated operators in general, but in the case when the underlying metric is flat or Einstein we shall present closed formulas for them.

Now let (M,h) be an n-dimensional Einstein manifold with normalization given by $Ric(h) = \frac{2J}{n}(n-1)h$ for a constant $J \in \mathbb{R}$. By [AG08], it follows that one can recover Branson-Gover operators as residues of solution operators, see Equation (3.15). More precisely, we have

$$\operatorname{Res}_{\lambda=\frac{\beta}{2}-N}\big(\mathcal{T}^{(+)}_{2N}(\lambda)\big) \sim (\tfrac{2J}{n})^N\Big[(\tfrac{\beta}{2}+N)R_N(y^{(+)};0) + (\tfrac{\beta}{2}-N)R^{(1)}_N(y^{(1)})\Big]. \tag{4.1}$$

The right hand side on the previous display is just the Branson-Gover operator of order $2N$, $N \in \mathbb{N}$, acting on differential p-forms:

$$L^{(p)}_{2N} = (\tfrac{2J}{n})^N\Big[(\tfrac{\beta}{2}+N)R_N(y^{(+)};0) + (\tfrac{\beta}{2}-N)R^{(1)}_N(y^{(1)})\Big]. \tag{4.2}$$

Note that there is no obstruction for its existence in even dimensions n. The normalization

$$\begin{aligned}\bar{L}^{(p)}_{2N} :&= (\tfrac{\beta}{2}-N+1)_{2N-1}L^{(p)}_{2N}\\ &= (\tfrac{2J}{n})^N\Big[(\tfrac{\beta}{2}-N+1)_{2N}R_N(y^{(+)};0) + (\tfrac{\beta}{2}-N)_{2N}R^{(1)}_N(y^{(1)})\Big]\end{aligned} \tag{4.3}$$

has the effect that the factors appearing in Theorem 4.3 are differential operators with polynomial coefficients.

Remark 4.1 The normalization factor $(\frac{\beta}{2}-N+1)_{2N-1}$ can vanish only in even dimensions n, due to $\beta = n-2p$. The zero locus is characterized by $l \in \{0,\dots,N-1\}$ such that $\frac{\beta}{2} = l$, or by the existence of $l \in \{1,\dots N-1\}$ such that $\frac{\beta}{2} = -l$. The last case can be excluded by choosing $p \leq \frac{n}{2}$.

Proposition 4.2 *Let $N \in \mathbb{N}$ and $p = 0,\dots,n$ when n is odd, and $N \in \mathbb{N}$ and $p = 0,\dots,n$ such that $(\frac{\beta}{2}-N+1)_{2N-1} \neq 0$ when n is even. The normalized Branson-Gover operators $\bar{L}^{(p)}_{2N}$ satisfy the recurrence relation*

$$\begin{aligned}\bar{L}^{(p)}_{2N} = \big[&(\tfrac{\beta}{2}+N)(\tfrac{\beta}{2}-N+1)\delta^h\mathrm{d} + (\tfrac{\beta}{2}+N-1)(\tfrac{\beta}{2}-N)\mathrm{d}\delta^h\\ &+ (\tfrac{\beta}{2}-N)(\tfrac{\beta}{2}-N+1)(\tfrac{\beta}{2}+N-1)(\tfrac{\beta}{2}+N)\tfrac{2J}{n}\big]\bar{L}^{(p)}_{2N-2},\end{aligned} \tag{4.4}$$

for $\bar{L}^{(p)}_0 := \mathrm{Id}$.

Proof. We shall use

$$\begin{aligned}\tfrac{2J}{n}R_m(y^{(+)};0) &= \big[\delta^h\mathrm{d} + (\tfrac{\beta}{2}+m-1)(\tfrac{\beta}{2}-m)\tfrac{2J}{n}\big]R_{m-1}(y^{(+)};0),\\ \tfrac{2J}{n}R^{(1)}_m(y^{(1)}) &= \big[\mathrm{d}\delta^h + (\tfrac{\beta}{2}+m)(\tfrac{\beta}{2}-m+1)\tfrac{2J}{n}\big]R^{(1)}_{m-1}(y^{(1)}) + (\tfrac{\beta}{2}-m+2)_{2m-2}\mathrm{d}\delta^h,\end{aligned}$$

where the first equality is easily verified and the second follows from Proposition 2.5. In addition, we need an elementary identity

$$\mathrm{d}\delta^h R_{N-1}(y^{(+)};0) = (\tfrac{\beta}{2}-N+1)_{2N-2}\mathrm{d}\delta^h.$$

We start with the evaluation of the right hand side of Equation (4.4). We have

$$
\begin{aligned}
&\Big[(\tfrac{\beta}{2}+N)(\tfrac{\beta}{2}-N+1)\delta^h\mathrm{d}+(\tfrac{\beta}{2}+N-1)(\tfrac{\beta}{2}-N)\mathrm{d}\delta^h \\
&\quad +(\tfrac{\beta}{2}-N)(\tfrac{\beta}{2}-N+1)(\tfrac{\beta}{2}+N-1)(\tfrac{\beta}{2}+N)\tfrac{2J}{n}\Big]\times \\
&\quad \times(\tfrac{2J}{n})^{N-1}\Big[(\tfrac{\beta}{2}-N+2)_{2N-2}R_{N-1}(y^{(+)};0)+(\tfrac{\beta}{2}-N+1)_{2N-2}R^{(1)}_{N-1}(y^{(1)})\Big] \\
&=(\tfrac{2J}{n})^{N-1}(\tfrac{\beta}{2}-N+1)_{2N}\big[\delta^h\mathrm{d}+(\tfrac{\beta}{2}-N)(\tfrac{\beta}{2}+N-1)\tfrac{2J}{n}\big]R_{N-1}(y^{(+)};0) \\
&\quad +(\tfrac{2J}{n})^{N-1}(\tfrac{\beta}{2}-N)_{2N}\big[\mathrm{d}\delta^h+(\tfrac{\beta}{2}-N+1)(\tfrac{\beta}{2}+N)\tfrac{2J}{n}\big]R^{(1)}_{N-1}(y^{(1)}) \\
&\quad +(\tfrac{2J}{n})^{N-1}(\tfrac{\beta}{2}-N)(\tfrac{\beta}{2}+N-1)(\tfrac{\beta}{2}-N+2)_{2N-2}\mathrm{d}\delta^hR_{N-1}(y^{(+)};0).
\end{aligned}
$$

The preparatory identities above ensure that this equals to

$$(\tfrac{2J}{n})^N\Big[(\tfrac{\beta}{2}-N+1)_{2N}R_N(y^{(+)};0)+(\tfrac{\beta}{2}-N)_{2N}R^{(1)}_N(y^{(1)})\Big]$$

and the proof is complete. □

The recurrence relation for $\bar{L}^{(p)}_{2N}$ implies part of the result [GŠ13, Theorem 5.3].

Theorem 4.3 *Let $N\in\mathbb{N}$ and $p=0,\dots,n$ when n is odd, and $N\in\mathbb{N}$ and $p=0,\dots,n$ such that $(\frac{\beta}{2}-N+1)_{2N-1}\neq 0$ when n is even. The normalized Branson-Gover operators $\bar{L}^{(p)}_{2N}$ factorize as*

$$
\begin{aligned}
\bar{L}^{(p)}_{2N}=\prod_{l=1}^{N}\big[&(\tfrac{\beta}{2}+N-l+1)(\tfrac{\beta}{2}-N+l)\delta^h\mathrm{d}+(\tfrac{\beta}{2}+N-l)(\tfrac{\beta}{2}-N+l-1)\mathrm{d}\delta^h \\
&+(\tfrac{\beta}{2}-N+l-1)(\tfrac{\beta}{2}-N+l)(\tfrac{\beta}{2}+N-l)(\tfrac{\beta}{2}+N-l+1)\tfrac{2J}{n}\big]. \qquad (4.5)
\end{aligned}
$$

In the setting of Theorem 4.3 we have

$$L^{(p)}_{2N}=\prod_{k=1}^{N}\big[\tfrac{\beta+2k}{\beta+2k-2}\delta^h\mathrm{d}+\tfrac{\beta-2k}{\beta-2k+2}\mathrm{d}\delta^h+(\tfrac{\beta}{2}-k)(\tfrac{\beta}{2}+k)\tfrac{2J}{n}\big], \qquad (4.6)$$

which holds in even dimensions n only for $\beta\notin\{0,2,\dots,2N-2\}$.

Now we discuss the exceptional cases when $(\frac{\beta}{2}-N+1)_{2N-1}=0$. By Remark 4.1 we have to consider $\beta=2l$ for $l\in\{0,\dots,N-1\}$. For these values the polynomials $R_N(y^{(+)};0)$ and $R^{(1)}_N(y^{(1)})$ factorize by $\delta^h\mathrm{d}$ and $\mathrm{d}\delta^h$, respectively, and this influences the factorization of $L^{(p)}_{2N}$.

Theorem 4.4 *Let n be even, $N\in\mathbb{N}$ and $p\leq\frac{n}{2}$.*

(1) *Let $\beta=2l$ for some $l\in\{1,\dots,N-1\}$. The Branson-Gover operator of order $2N$ factorizes by*

$$L^{(p)}_{2N}=-\tfrac{2l}{2l-1}\widetilde{P}^{(p)}\times\prod_{k=1,k\neq l,l+1}^{N}\big[\tfrac{l+k}{l+k-1}\delta^h\mathrm{d}+\tfrac{l-k}{l-k+1}\mathrm{d}\delta^h+(l-k)(l+k)\tfrac{2J}{n}\big], \qquad (4.7)$$

where

$$\tilde{P}^{(p)} := \left[\tfrac{2l+1}{2}\delta^h\mathrm{d} + \tfrac{2l-1}{2}\mathrm{d}\delta^h\right]\left[\mathrm{d}\delta^h - \delta^h\mathrm{d} + 2l\tfrac{2J}{n}\right]$$

is a fourth-order differential operator.

(2) *Let $\beta = 0$. The Branson-Gover operator of order $2N$ factorizes by*

$$L^{(p)}_{2N} = N\left[\delta^h\mathrm{d} - \mathrm{d}\delta^h\right] \times \prod_{k=2}^{N}\left[\delta^h\mathrm{d} + \mathrm{d}\delta^h - k(k-1)\tfrac{2J}{n}\right]. \tag{4.8}$$

Proof. First note that $\tilde{P}^{(p)}$ decomposes, due to $\mathrm{d}^2 = 0 = (\delta^h)^2$, as

$$\tilde{P}^{(p)} = -\tfrac{2l+1}{2}\delta^h\mathrm{d}\left[\delta^h\mathrm{d} - 2l\tfrac{2J}{n}\right] + \tfrac{2l-1}{2}\mathrm{d}\delta^h\left[\mathrm{d}\delta^h + 2l\tfrac{2J}{n}\right] =: \widetilde{P}^{(p)}_1 + \widetilde{P}^{(p)}_2.$$

For $l \in \{1, \ldots, N-1\}$ such that $\beta = 2l$ the polynomials $R_m(y^{(+)}; 0)$ and $R^{(1)}_m(y^{(1)})$, cf. Equation (2.1) and (2.6), satisfy by Proposition 2.4

$$\begin{aligned}
R_m(y^{(+)}; 0) &= -\tfrac{2}{2l+1}(\tfrac{n}{2J})^2\widetilde{P}^{(p)}_1 \times \prod_{k=1,k\neq l,l+1}^{N}\left[\tfrac{n}{2J}\delta^h\mathrm{d} + (l-k)(l+k-1)\right],\\
R^{(1)}_m(y^{(1)}) &= R_m(y^{(1)}; 0) + 0\\
&= \tfrac{2}{2l-1}(\tfrac{n}{2J})^2\widetilde{P}^{(p)}_2 \prod_{k=1,k\neq l,l+1}^{N}\left[\tfrac{n}{2J}\mathrm{d}\delta^h + (l+k)(l-k+1)\right].
\end{aligned}$$

Since $\widetilde{P}^{(p)}_1$ and $\widetilde{P}^{(p)}_2$ factorize by $\delta^h\mathrm{d}$ and $\mathrm{d}\delta^h$, respectively, the last display is equivalent to

$$\begin{aligned}
R_m(y^{(+)}; 0) &= -\tfrac{2l}{2l-1}\tfrac{1}{l+N}(\tfrac{n}{2J})^N\widetilde{P}^{(p)}_1 \times\\
&\quad\times \prod_{k=1,k\neq l,l+1}^{N}\left[\tfrac{l+k}{l+k-1}\delta^h\mathrm{d} + \tfrac{l-k}{l-k+1}\mathrm{d}\delta^h + (l-k)(l+k)\tfrac{2J}{n}\right],\\
R^{(1)}_m(y^{(1)}) &= -\tfrac{2l}{2l-1}\tfrac{1}{l-N}(\tfrac{n}{2J})^N\widetilde{P}^{(p)}_2 \times\\
&\quad\times \prod_{k=1,k\neq l,l+1}^{N}\left[\tfrac{l-k}{l-k+1}\mathrm{d}\delta^h + \tfrac{l+k}{l+k-1}\delta^h\mathrm{d} + (l-k)(l+k)\tfrac{2J}{n}\right].
\end{aligned}$$

Hence, the definition of $L^{(p)}_{2N}$, see (4.2), gives

$$L^{(p)}_{2N} = -\tfrac{2l}{2l-1}\widetilde{P}^{(p)} \times \prod_{k=1,k\neq l,l+1}^{N}\left[\tfrac{l-k}{l-k+1}\mathrm{d}\delta^h + \tfrac{l+k}{l+k-1}\delta^h\mathrm{d} + (l-k)(l+k)\tfrac{2J}{n}\right],$$

which proves the first claim.

For $\beta = 0$, the polynomials $R_m(y^{(+)}; 0)$ and $R^{(1)}_m(y^{(1)})$ (cf. Equation (2.1) and (2.6)) satisfy by Proposition 2.4

$$\begin{aligned}
R_m(y^{(+)}; 0) &= \tfrac{n}{2J}\delta^h\mathrm{d}\prod_{k=2}^{N}\left[\tfrac{n}{2J}\delta^h\mathrm{d} - k(k-1)\right],\\
R^{(1)}_m(y^{(1)}) &= R_m(y^{(1)}; 0) + 0
\end{aligned}$$

$$= \tfrac{n}{2J}\mathrm{d}\delta^h \prod_{k=2}^{N} \big[\tfrac{n}{2J}\mathrm{d}\delta^h - k(k-1)\big].$$

Since they factorize by $\delta^h\mathrm{d}$ and $\mathrm{d}\delta^h$, respectively, we can write

$$R_m(y^{(+)};0) = (\tfrac{n}{2J})^N \delta^h\mathrm{d} \prod_{k=2}^{N} \big[\delta^h\mathrm{d} + \mathrm{d}\delta^h - k(k-1)\tfrac{2J}{n}\big],$$

$$R_m^{(1)}(y^{(1)}) = (\tfrac{n}{2J})^N \mathrm{d}\delta^h \prod_{k=2}^{N} \big[\mathrm{d}\delta^h + \delta^h\mathrm{d} - k(k-1)\tfrac{2J}{n}\big].$$

Hence the definition of $L^{(p)}_{2N}$, see (4.2), gives

$$L^{(p)}_{2N} = N\big[\delta^h\mathrm{d} - \mathrm{d}\delta^h\big] \times \prod_{k=2}^{N} \big[\delta^h\mathrm{d} + \mathrm{d}\delta^h - k(k-1)\tfrac{2J}{n}\big]$$

and the proof is complete. □

From now on let n be even. We proceed with explicit formulas for the critical Branson-Gover operator, gauge companion operator and Q-curvature operator:

Proposition 4.5 *The critical Branson-Gover operator $L^{(p)}_{n-2p}$ is given by the product formula*

$$L^{(p)}_{n-2p} = (n-2p)\delta^h\mathrm{d} \circ \prod_{l=1}^{\frac{n-2p}{2}-1} \big[\delta^h\mathrm{d} + (\tfrac{\beta}{2} - l)(\tfrac{\beta}{2} + l - 1)(\tfrac{2J}{n})\big]. \tag{4.9}$$

Proof. It follows from Equation (4.1) that

$$L^{(p)}_{n-2p} = (n-2p)(\tfrac{2J}{n})^{\frac{n-2p}{2}} R_{\frac{n-2p}{2}}(y^{(+)};0)$$

$$= (n-2p) \prod_{l=1}^{\frac{n-2p}{2}} \big[\delta^h\mathrm{d} + (\tfrac{\beta}{2} - l)(\tfrac{\beta}{2} + l - 1)\tfrac{2J}{n}\big].$$

Note that the last factor reduces to

$$\delta^h\mathrm{d} + (\tfrac{\beta}{2} - \tfrac{n-2p}{2})(\tfrac{\beta}{2} + \tfrac{n-2p}{2} - 1)\tfrac{2J}{n} = \delta^h\mathrm{d},$$

since $\beta = n-2p$. This completes the proof. □

Consequently, we found the explicit formulas for the Q-curvature operator

$$Q^{(p+1)}_{n-2p-2} = (n-2p) \prod_{l=1}^{\frac{n-2p}{2}-1} \big[\delta^h\mathrm{d} + (\tfrac{\beta}{2} - l)(\tfrac{\beta}{2} + l - 1)(\tfrac{2J}{n})\big]$$

and the gauge companion operator

$$G^{(p+1)}_{n-2p-1} = (n-2p)\delta^h \circ \prod_{l=1}^{\frac{n-2p}{2}-1} \left[\delta^h \mathrm{d} + (\tfrac{\beta}{2}-l)(\tfrac{\beta}{2}+l-1)(\tfrac{2J}{n})\right].$$

Obviously,

$$L^{(p)}_{n-2p} = G^{(p+1)}_{n-2p-1} \circ \mathrm{d} = \delta^h \circ Q^{(p+1)}_{n-2p-2} \circ \mathrm{d},$$

which is the famous double factorization of the critical Branson-Gover operator.

Remark 4.6 Let $(M,h) = (\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ be the euclidean space. The explicit formulas for $L^{(p)}_{2N}$, $Q^{(p+1)}_{n-2p-2}$ and $G^{(p+1)}_{n-2p-1}$ immediately imply after setting $J=0$ that

$$\begin{aligned} L^{(p)}_{2N} &= (\tfrac{n-2p}{2}+N)(\delta^h \mathrm{d})^N + (\tfrac{n-2p}{2}-N)(\mathrm{d}\delta^h)^N, \\ Q^{(p+1)}_{n-2p-2} &= (n-2p)(\mathrm{d}\delta^h)^{\frac{n-2p}{2}-1}, \\ G^{(p+1)}_{n-2p-1} &= (n-2p)\delta^h(\mathrm{d}\delta^h)^{\frac{n-2p}{2}-1}. \end{aligned}$$

Appendix A. Generalized Hypergeometric functions and dual Hahn polynomials

Here we summarize a few basic conventions and definitions related to generalized hypergeometric functions.

The Pochhammer symbol of a complex number $a \in \mathbb{C}$ is defined by $(a)_m := \frac{\Gamma(a+m)}{\Gamma(a)}$, for $m \in \mathbb{Z}$ and $(a)_0 := 1$. If m is a positive integer, $(a)_m = a(a+1)\cdots(a+m-1)$.

The generalized hypergeometric function ${}_pF_q$ of type (p,q), $p,q \in \mathbb{N}$, is defined by

$${}_pF_q \left[\begin{matrix} a_1 \ , \ \dots \ , \ a_p \\ b_1 \ , \ \dots \ , \ b_q \end{matrix}; z\right] := \sum_{l=0}^{\infty} \frac{(a_1)_l \dots (a_p)_l}{(b_1)_l \dots (b_q)_l} \frac{z^l}{l!}, \tag{A.1}$$

for $a_i \in \mathbb{C}$ $(1 \le i \le q)$, $b_j \in \mathbb{C} \setminus \{-\mathbb{N}_0\}$ $(1 \le j \le q)$, and $z \in \mathbb{C}$.

The dual Hahn polynomials [KM61] are defined by

$$R_m\big(\lambda(n); a, b, N\big) = {}_3F_2 \left[\begin{matrix} -m \ , \ -n \ , \ n+a+b+1 \\ a+1 \ , \ -N+1 \end{matrix}; 1\right]$$

for $n, N \in \mathbb{N}$, $m = 0, \dots, N-1$, $a, b \in \mathbb{C}$ such that the real parts fulfill $\mathfrak{R}(a), \mathfrak{R}(b) > -1$ and $\lambda(n) := n(n+a+b+1)$.

Here we prove the two identities for collections of $C^{(+)}_k(m)$ and $D^{(1)}_k(m)$, defined in Equations (2.3) and (2.9):

Lemma A.1. *We have*

$$\sum_{k=0}^{m} C^{(+)}_k(m)(\tfrac{\beta}{2}-k)_{2k} = (\lambda)_m(\tfrac{\beta}{2})_m,$$

$$\sum_{k=0}^{m} D^{(1)}_k(m)(\tfrac{\beta}{2}-k+1)_{2k} = (\lambda)_m(\tfrac{\beta}{2})_m(\lambda-\beta).$$

Proof. By standard Pochhammer identities, we rewrite the sums as

$$\sum_{k=0}^{m} C_k^{(+)}(m)(\tfrac{\beta}{2}-k)_{2k} = (\lambda-\tfrac{\beta}{2}+1)_m(\tfrac{\beta}{2})_m \sum_{k=0}^{m} \frac{(-m)_k(-\frac{\beta}{2}+1)_k}{k!(\lambda-\frac{\beta}{2}+1)_k}$$
$$= (\lambda-\tfrac{\beta}{2}+1)_m(\tfrac{\beta}{2})_m \times {}_2F_1(-m,-\tfrac{\beta}{2}+1;\lambda-\tfrac{\beta}{2}+1;1),$$
$$\sum_{k=0}^{m} D_k^{(1)}(m)(\tfrac{\beta}{2}-k+1)_{2k} = (\lambda-\tfrac{\beta}{2}+1)_m(\tfrac{\beta}{2}+1)_{m-1} \sum_{k=0}^{m} \frac{(-m)_k(-\frac{\beta}{2})_k}{k!(\lambda-\frac{\beta}{2}+1)_k}\times$$
$$\times \left[k(\lambda+\beta+2m)+\tfrac{\beta}{2}(\lambda-\beta-2m)\right]$$
$$= (\lambda-\tfrac{\beta}{2}+1)_m(\tfrac{\beta}{2}+1)_{m-1}\Big[\frac{m\frac{\beta}{2}(\lambda+\beta+2m)}{\lambda-\frac{\beta}{2}+1} \times {}_2F_1(-m+1,-\tfrac{\beta}{2}+1;\lambda-\tfrac{\beta}{2}+2;1)$$
$$+ \tfrac{\beta}{2}(\lambda-\beta-2m) \times {}_2F_1(-m,-\tfrac{\beta}{2};\lambda-\tfrac{\beta}{2}+1;1)\Big].$$

Applying the Chu-Vandermonde identity,

$${}_2F_1\left[\begin{matrix}-m,\alpha\\ \gamma\end{matrix};1\right] = \frac{(\gamma-\alpha)_m}{(\gamma)_m}$$

with $m \in \mathbb{N}$ and appropriate $\alpha, \gamma \in \mathbb{C}$, the first claim follows. The proof of the second statement is based on the identity

$$m(\lambda+\beta+2m)(\lambda+1)_{m-1} + (\lambda-\beta-2m)(\lambda+1)_m = (\lambda)_m(\lambda-\beta).$$

This completes the proof. $\square$

Appendix B. Poincaré-Einstein metric construction

Here we briefly review the content of Poincaré-Einstein metric construction, [FG11]. Let (M^n, h) be an n-dimensional semi-Riemannian manifold, $n \geq 3$. On $X := M \times (0, \varepsilon)$ for $\varepsilon > 0$, we consider the metric

$$g_+ = r^{-2}(\mathrm{d}r^2 + h_r),$$

for a 1-parameter family of metrics h_r on M such that $h_0 = h$. The requirement of Einstein condition on g_+ for n odd,

$$Ric(g_+) + ng_+ = O(r^\infty),$$

uniquely determines the family h_r, while for n even the conditions

$$Ric(g_+) + ng_+ = O(r^{n-2}),$$
$$\mathrm{tr}(Ric(g_+) + ng_+) = O(r^{n-1}),$$

uniquely determine the coefficients $h_{(2)}, \dots, h_{(n-2)}$, $\tilde{h}_{(n)}$ and the trace of $h_{(n)}$ in the formal power series

$$h_r = h + r^2 h_{(2)} + \cdots + r^{n-2} h_{(n-2)} + r^n(h_{(n)} + \tilde{h}_{(n)} \log r) + \cdots \quad .$$

For example, we have

$$h_{(2)} = -P, \quad h_{(4)} = \frac{1}{4}(P^2 - \frac{B}{n-4}),$$

where P is the Schouten tensor and B is the Bach tensor associated to h. The metric g_+ on X is called Poincaré-Einstein metric associated to the semi-Riemannian manifold (M,h).

Two different representatives $h,\widehat{h} \in [h]$ in the conformal class lead to Poincaré-Einstein metrics g_+^1 and g_+^2 related by a diffeomorphism $\Phi : U_1 \subset X \to U_2 \subset X$, where both U_i, $i=1,2$, contain $M \times \{0\}$, $\Phi|_M = \mathrm{Id}_M$ and $g_+^1 = \Phi^* g_+^2$ (up to a finite order in r for even n).

The explicit knowledge of h_r is available in a few cases, e.g., [FG11, LN10]. If (M,h) is the euclidean space, one can realize the Poincaré-Einstein metric as the hyperbolic metric on the upper half space, while if (M,h) is an Einstein manifold normalized by $Ric(h) = \frac{2J}{n}(n-1)h$ for some constant $J \in \mathbb{R}$, the 1-parameter family of metrics h_r is

$$h_r = (1 - \tfrac{J}{2n} r^2)^2 h.$$

In both cases, there are no obstructions in even dimensions.

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E. Čech Institute, Mathematical Institute of Charles University, Sokolovská 83, Praha 8 - Karlín, Czech Republic

E-mail address: `fischmann@karlin.mff.cuni.cz, somberg@karlin.mff.cuni.cz`